The Stieltjes and Hausdorff Moment Problems

Abstract

We discuss classical moment problems, focusing on the Stieltjes and Hausdorff cases. Given a sequence of real numbers, the central question is to determine whether it can be represented as moments of a positive measure supported on a prescribed domain. In the Stieltjes moment problem, the measure is supported on $[0, \infty)$, while in the Hausdorff moment problem, the support is restricted to $[0, 1]$. We present fundamental characterizations of such sequences in terms of positivity of Hankel matrices and complete monotonicity conditions, respectively. Particular emphasis is placed on the contrast between the two problems, especially regarding existence and uniqueness of representing measures. We also illustrate these ideas through examples and briefly indicate connections with functional analysis and operator theory.

1 Introduction

Moment problems arise naturally in analysis and probability theory, where one seeks to represent a sequence as moments of a measure. Given a sequence $\{m_n\}_{n=0}^{\infty}$, the aim is to determine whether there exists a positive Borel measure μ such that

$$m_n = \int t^n \, d\mu(t), \quad \forall n \geq 0.$$

2 Stieltjes Moment Problem

The Stieltjes moment problem asks for such a representation when the measure is supported on $[0, \infty)$. A sequence $\{m_n\}$ is called a Stieltjes moment sequence if there exists a positive Borel measure μ on $[0, \infty)$ such that

$$m_n = \int_0^{\infty} t^n \, d\mu(t).$$

A key characterization involves the positivity of Hankel matrices:

$$(m_{i+j})_{i,j=0}^N, \quad (m_{i+j+1})_{i,j=0}^N$$

for all $N \in \mathbb{N}$.

3 Hausdorff Moment Problem

The Hausdorff moment problem concerns measures supported on $[0, 1]$. A sequence $\{m_n\}$ is called a Hausdorff moment sequence if there exists a positive Borel measure μ on $[0, 1]$ such that

$$m_n = \int_0^1 t^n \, d\mu(t).$$

Such sequences are completely characterized by complete monotonicity:

$$(-1)^k \Delta^k m_n \geq 0, \quad \forall n, k \geq 0,$$

where $\Delta m_n = m_{n+1} - m_n$.

4 Conclusion

The Hausdorff moment problem is always determinate, whereas the Stieltjes moment problem may admit multiple representing measures. These problems illustrate deep connections between sequences, measures, and positivity, and have important applications in functional analysis and operator theory.