Kinematic Lie algebras

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1 Abstract

Ever since Minkowski introduced his eponymous spacetime, lorentzian geometry has played a fundamental rôle in Physics and, until recently, has dominated our attempts to model the universe. Minkowski spacetime replaced the Galilei spacetime of newtonian mechanics as the geometric arena for Einstein's theory of special relativity, conceived to describe Maxwell's theory of electrodynamics. With the advent of quantum mechanics, it became the arena of relativistic quantum mechanics and, unavoidably, of quantum field theory. On the other hand, Minkowski spacetime is the flat model of lorentzian geometry, the Cartan geometry modelled on it, and which is the basis of Einstein's theory of general relativity, which accurately describes a wide range of gravitational phenomena. The attempt at marrying quantum theory and general relativity into a quantum theory of gravity has kept the theoretical/mathematical physics community busy for the best part of the last 75 years. Why then should one bother with non-lorentzian spacetimes, except as a purely mathematical curiosity? One answer to this question lies precisely in the difficulty to formulate a quantum theory of gravity. It may help to keep the following picture in mind: the Bronstein cube of physical theories. This cube is a cartoon of physical theories one can obtain from classical mechanics (CM) by turning on certain parameters: the inverse speed of light 1/c, Newton's gravitational constant (or, more geometrically, curvature) G and Planck's constant h. Of course, these are physical constants and as such they take particular values, but let us pretend that we can change them at will. There are three directions we can go in from classical mechanics: to special relativity (SR) by turning on 1/c, to quantum mechanics (QM) by turning on h, and to newtonian gravity (NG) by turning on G. From special relativity we may go to general relativity (GR) by turning on G and to relativistic quantum mechanics and hence quantum field theory (QFT) by turning on h. These two theories are the standard points of departure towards the final goal of a theory of quantum gravity (QG). However as the picture makes clear, there is a third possible line of approach: via the (as yet non-existent) quantisation of newtonian gravity (QNG). In this review I shall remain in the non-quantum world and concentrate on the bottom side of the Bronstein cube. In geometrical terms, clas- QNG sical mechanics and special relativity are described by Klein geometries: Galilei and Minkowski spacetimes, whereas

turning on G corresponds to constructing Cartan geometries modelled on them. I shall therefore start by describing Galilei and Minkowski spacetimes and show that they are Klein geometries of the Galilei and Poincaré groups, respectively. I shall recognise these groups as examples of kinematical Lie groups and recall the classification of kinematical Lie algebras. This will lead to the classi- fication of kinematical Klein geometries, of which Galilei and Minkowski spacetimes are but two of a plethora of examples which nevertheless give rise to a small class of Cartan geometries of spacetimes: lorentzian, ga- lilean (or Newton–Cartan), carrollian and aristotelian. I do not discuss aristotelian geometries in this review and concentrate on the galilean and carrollian geometries.